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Global Weak Solutions of the Boltzmann Equation

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A new definition of the concept of weak solution of the nonlinear Boltzmann equation is introduced. It is proved that, without any truncation on the collision kernel, the Boltzmann equation in the one-dimensional case has a global weak solution in this sense. Global conservation of energy follows.

KEY WORDS: Boltzmann equation; energy conservation; global solution.

1. INTRODUCTION

We are concerned with the initial value problem for the nonlinear Boltzmann equation when the solution depends on just one space coordinate which might range from $-\infty$ to $+\infty$ or from 0 to 1 (with periodicity boundary conditions); for definiteness we stick to the latter case. Easy modifications, in the vein of ref. 4, are necessary to deal with the case of different boundary conditions. The x-, y- and z- component of the velocity $\mathbf{v} \in \mathbf{R}^3$ will be denoted by ξ, η and ζ, respectively, and the equation reads

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q(f, f) \tag{1.1}$$

with

$$Q(f, f)(x, \mathbf{v}, t) = \int \int B(\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*), |\mathbf{v} - \mathbf{v}_*|)\, (f' f'_* - f f_*) \sin\theta \, d\theta \, d\phi \, d\mathbf{v}_*. \tag{1.2}$$

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For a detailed explanation of the structure of the collision term, see ref. 3, 5, or 6. The angles θ and ϕ are the polar and azimuthal angles of the collision parameter $\mathbf{n}\in S^2$ relative to a polar axis in direction $\mathbf{V}=\mathbf{v}-\mathbf{v}_*$.

We remark that if we assume, as in ref. 2, that there is an $\epsilon>0$ such that

$$B(\ldots)=0 \quad \text{if } |(\mathbf{v}-\mathbf{v}_*)\cdot\mathbf{n}|\leq\sqrt{\epsilon} \tag{1.3a}$$

and that

$$B \text{ is uniformly bounded.} \tag{1.3b}$$

Then there is a weak solution in the traditional sense of the Boltzmann equation (2). Here we introduce a motivated new definition of weak solution and show that the truncation is not needed.

To this end we introduce what we call the weak form of the collision term, which we still denote by $Q(f,f)$. We shall henceforth use the latter notation for the operator defined by

$$\begin{aligned}
&\int_{[0,T]\times[0,1]\times\mathbf{R}^3} Q(f,f)(x,\mathbf{v},t)\varphi(x,\mathbf{v},t)d\mathbf{v}\,dx\,dt\\
&\quad=\frac{1}{2b}\int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3\times S^2} B(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_*),|\mathbf{v}-\mathbf{v}_*|)\\
&\quad\times(\varphi'+\varphi'_*-\varphi-\varphi_*)ff_*\,d\mu\,dt,
\end{aligned} \tag{1.4}$$

for any test function $\varphi(x,\mathbf{v},t)$, which is twice differentiable as a function of $\mathbf{v}$ with second derivatives uniformly bounded with respect to x and t. In Eq. (1.4) we have used the notation

$$d\mu=\sin\theta\,d\theta\,d\phi\,d\mathbf{v}_*\,d\mathbf{v}\,dx.$$

We remark that for classical solutions the above definition is known to be equivalent to that in (1.2). The main reason for introducing it is that it may produce weak solutions (as opposed to renormalized solutions in the sense of DiPerna and Lions (6)) even if the collision term is not necessarily in L^1. This also avoids cutting off the small relative speeds, as done in ref. 4.

Another advantage is that we might, in principle, consider solutions for inverse power potentials without introducing Grad's angular cutoff, although we shall not attempt this in the present paper. On the other hand we shall not allow a growth for large values of $|\mathbf{V}|$, i.e., we exclude in this paper hard spheres and potentials harder than the inverse fifth power. This is an important technical simplification, which perhaps might be removed by much harder work.

For a function f to be a weak solution of the Boltzmann equation, it must satisfy Eq. (1.1), where the derivatives in the left hand side are distributional derivatives and the right hand side has been defined above.

We have an initial value $f(x, \mathbf{v}, 0) = f_0(x, \mathbf{v})$, and we shall assume that $f_0 \in L^1_+([0, 1] \times \mathbf{R}^3)$ with the normalization

$$\int\int f_0 \, dx \, d\mathbf{v} = 1. \tag{1.5}$$

The association of the solution with the weak formulation is standard.

The objective of this paper is to show that the initial value problem for the Boltzmann equation has a global weak solution in the sense defined above. The main step in proving this is a proof that collision term $Q(f, f)$ is such that the expression in Eq. (1.4) is finite.

2. BASIC ESTIMATES

We now set out to prove the crucial estimates for the solution of the initial value problem and for the collision term. It is safe to assume that we deal with a sufficiently regular solution of the problem, because this can always be enforced by truncating the collision kernel and modifying the collision terms in the way described in earlier work, in particular in ref. 6. If we obtain strong enough bounds on the solutions of such truncated problems, we can then extract a subsequence converging to a renormalized solution in the sense of DiPerna and Lions; and the bounds which we do get actually guarantee that this solution is then a solution in the weak sense defined above.

We need some additional notation. For each $x \in [0, 1]$ and $t \geqslant 0$, let

$$\begin{aligned}
\rho(x,t) &= \int f(x, \mathbf{v}, t) \, d\mathbf{v}, \\
m(t) &= \int \rho(x,t) \, dx, \\
j(x,t) &= \int \xi f(x, \mathbf{v}, t) \, d\mathbf{v}, \\
p(x,t) &= \int \xi^2 f(x, \mathbf{v}, t) \, d\mathbf{v}, \\
q(x,t) &= \int \xi \, |\mathbf{v}|^2 f(x, \mathbf{v}, t) \, d\mathbf{v}.
\end{aligned} \tag{2.1}$$

We call ρ the mass density, $m(t)$ the total mass, j the mass flux (or momentum) in x-direction, p the momentum flux, and q the energy flux. We shall also need the total energy, defined by

$$E(t) = \int_0^1 \int |\mathbf{v}|^2 f \, d\mathbf{v} \, dx.$$

Constants are denoted by C, but C can denote a different constant in different formulas.

Consider now the functional

$$I[f](t) = \underbrace{\int\int}_{x<y} \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*) f(x, \mathbf{v}, t) f(y, \mathbf{v}_*, t) \, d\mathbf{v}_* \, d\mathbf{v} \, dx \, dy, \tag{2.2}$$

where the first double integral is over the triangle $0 \leqslant x < y \leqslant 1$. This functional was in the one-dimensional discrete velocity context first introduced by Bony (1). In the case without boundaries, it has become known as "potential for interaction". The use of this functional is the main reason why we have to restrict our work to one dimension. No functional with similar pleasant properties is known, at this time, in more than one dimension. Notice that if we have bounds for $\int_0^1 \rho(x, t) \, dx$ (which is conserved by (1.8b)) and for $\int_0^1 j(x, t) \, dx$, then we have control over the functional $I[f](t)$.

A short calculation with proper use of the collision invariants of the Boltzmann collision operator shows that

$$\frac{d}{dt} I[f] = -\int_{[0,1]} \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, t) f(x, \mathbf{v}, t) \, d\mathbf{v} \, d\mathbf{v}_* \, dx. \tag{2.3}$$

Notice that the first term on the right, apart from the factor $(\xi - \xi_*)^2$, has structural similarity to the collision term of the Boltzmann equation, and the integrand is nonnegative. This is the reason why the functional $I[f]$ is a powerful tool.

After integration from 0 to $T > 0$ and reorganizing,

$$\begin{aligned} &\int_0^T \int_{[0,1]} \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, t) f(x, \mathbf{v}, t) \, d\mathbf{v} \, d\mathbf{v}_* \, dx \, dt \\ &\quad = I[f](0) - I[f](T). \end{aligned} \tag{2.4}$$

According to a previous remark, the left-hand side of (2.5) is bounded. Since the total energy is conserved, we have proved.

Lemma 2.1. If f is a sufficiently smooth solution of the initial value problem given by (1.1) and (1.4) with initial value f_0, then

$$E(t)$$

and

$$\int_0^t \int_0^1 \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, \tau) f(x, \mathbf{v}, \tau)\, d\mathbf{v}\, d\mathbf{v}_*\, dx\, d\tau$$

are bounded.

The idea of the basic estimates was given in ref. 2; we will repeat some details here to make this paper self-contained.

First, we formulate the end result as a Lemma.

Lemma 2.2. If the solution of the initial value problem for (1.1), (1.4) exists as a classical solution for $t \in (0, T)$, and if the initial value f_0 has a finite energy $E(0) = \int_0^1 \int |\mathbf{v}|^2 f_0(x, \mathbf{v})\, d\mathbf{v}\, dx$, then the expression in (1.4) is bounded in terms of constants depending on the initial data only for any test function $\varphi(x, \mathbf{v}, t)$ which is twice differentiable as a function of $\mathbf{v}$ with second derivatives uniformly bounded with respect to x and t.

As a corollary of the proof of Lemma 2.1.

Lemma 2.3. Let u_1 be the x-component of the bulk velocity

$$u_1 = \frac{\int \xi f\, d\mathbf{v}}{\int f\, d\mathbf{v}}. \tag{2.5}$$

Then

$$\begin{aligned} &\int_{[0,1]\times\mathbf{R}^3\times[\tau,T]\times\mathbf{R}} (\xi - u_1)^2 f(x, \mathbf{v}, t) f(x, , \mathbf{v}_{*,} t) dx\, dt\, d\mathbf{v}\, d\mathbf{v}_* \\ &\qquad < K_0 \quad (0 < \tau < T), \end{aligned} \tag{2.6}$$

where K_0 is a constant, which only depends on the initial data.

In fact, Eq. (2.6) is nothing else than the left-hand side of Eq. (2.4) suitably rearranged. It is enough to expand the squares in both Eqs. (2.4) and (2.6), and replace $\int \xi f\, d\mathbf{v}$ by $u_1 \int f\, d\mathbf{v}$, according to Eq. (2.6), in the former equation, to obtain the latter (with $2K_0$ in place of K_0).

We have now the following Lemma 2.4.

Lemma 2.4. Under the above assumptions and the additional assumption that the ratio r between $\int_{\mathcal{S}}[\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_*)]^2 B(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}*), |\mathbf{v}-\mathbf{v}_*|)\sin\theta\, d\theta\, d\phi$ and $|\mathbf{v}-\mathbf{v}_*|^2 \int_{\mathcal{S}} B(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}*), |\mathbf{v}-\mathbf{v}_*|)\sin\theta\, d\theta\, d\phi$ is bounded from below, we have, for smooth solutions:

$$\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2\times[0,T]\times[0,1]} |\mathbf{v}-\mathbf{v}_*|^2 f(x,\mathbf{v},t) f(x,\mathbf{v}_*,t) B(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}*), |\mathbf{v}-\mathbf{v}_*|) dt\, d\mu < K_0, \tag{2.7}$$

where K_0 is a constant, which only depends on the initial data.

In fact, we can multiply the Boltzmann equation by ξ^2 and integrate with respect to $\mathbf{v}$, x, t. We can now replace ξ^2 by $(\xi-u_1)^2$ in the right hand side (since the extra terms vanish thanks to mass and momentum conservation) and after that separate the loss and gain terms. The loss term is bounded because of (2.6) (please remember that $B(|\mathbf{v}-\mathbf{v}_*|,\mathbf{n})$ is bounded). The gain term will be bounded because the left hand side is bounded (energy is bounded) and the loss term is bounded. But the gain term (using $c_1=\xi-u_1$) is given by

$$\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2\times[0,T]\times[0,1]} \{c_1^2 - 2n_1 c_1 \mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_*) + [\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_*)]^2\} f(x,\mathbf{v},t) f(x,\mathbf{v}_*,t) B(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}*), |\mathbf{v}-\mathbf{v}_*|) dt\, d\mu. \tag{2.8}$$

Now the first two contributions to the integral (coming from c_1^2 and $-2n_1c_1(\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_*)$ are bounded, because of (2.6) (please note that the integrals of n_1n_2 and n_1n_3 with respect to the angular variables vanish). Then, we conclude that the third one is bounded as well. Because of the assumption on the ratio r, the lemma follows.

We want to show that if $\psi(\mathbf{v})\in C_2(-\infty,\infty)$, with $|\partial^2\psi/\partial v_i\partial v_k| < K$ ($K=$ const.), the weak form of the collision operator

$$\int_{R^3} \psi(\mathbf{v}) Q(f,f)(\mathbf{v}) d(\mathbf{v}) = \frac{1}{2}\int_{R^3}\int_{R^3}\int_{\Omega^2} ff(\mathbf{v})[\psi(\mathbf{v}')+\psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v})] B(|\mathbf{v}_*-\mathbf{v}|,\theta) d\mathbf{v}\, d\mathbf{v}_*\, d\mathbf{n}$$

satisfies

$$\left|\int_{R^3} \psi(\mathbf{v}) Q(f,f)(\mathbf{v}) d(\mathbf{v})\right| \le C\int_{R^3}\int_{R^3}\int_{\Omega^2} f(\mathbf{v}_*) f(\mathbf{v}) |\mathbf{v}_*-\mathbf{v}|^2 B(|\mathbf{v}_*-\mathbf{v}|,\theta) d\mathbf{v}\, d\mathbf{v}_*\, d\mathbf{n}.$$

The result follows from the Taylor formula

$$\psi(\mathbf{v}'_*) - \psi(\mathbf{v}) = \sum_{i=1}^{3} \frac{\partial \psi}{\partial v_i}(v'_{*i} - v_i) + \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]'_* (v'_{*i} - v_i)(v'_{*k} - v_k),$$

$$\psi(\mathbf{v}_*) - \psi(\mathbf{v}) = \sum_{i=1}^{3} \frac{\partial \psi}{\partial v_i}(v_{*i} - v_i) + \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]_* (v_{*i} - v_i)(v_{*k} - v_k),$$

$$\psi(\mathbf{v}') - \psi(\mathbf{v}) = \sum_{i=1}^{3} \frac{\partial \psi}{\partial v_i}(v'_i - v_i) + \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]' (v'_i - v_i)(v'_k - v_k),$$

where the first derivatives are evaluated at $\mathbf{v}$ and the second derivatives at different points in velocity space (as indicated by the labels). Then

$$\begin{aligned}\psi(\mathbf{v}') + \psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v}) &= \sum_{i=1}^{3} \frac{\partial \psi}{\partial v_i}(v'_i - v_i + v'_{*i} - v_{*i}) \\ &+ \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]_* (v_{*i} - v_i)(v_{*k} - v_k) \\ &+ \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]'_* (v'_{*i} - v_i)(v'_{*k} - v_k) \\ &- \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]' (v'_i - v_i)(v'_k - v_k).\end{aligned}$$

The expression multiplying the first derivatives is zero because of momentum conservation; hence

$$\begin{aligned}\psi(\mathbf{v}') + \psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v}) &= \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]_* (v_{*i} - v_i)(v_{*k} - v_k) \\ &+ \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]'_* (v'_{*i} - v_i)(v'_{*k} - v_k) - \sum_{i,k=1}^{3} \left[\frac{\partial^2 \psi}{\partial v_i \partial v_k}\right]' (v'_i - v_i)(v'_k - v_k)\end{aligned}$$

and

$$|\psi(\mathbf{v}') + \psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v})| \leqslant 9K(|\mathbf{v}_* - \mathbf{v}|^2 + |\mathbf{v}'_* - \mathbf{v}|^2 + |\mathbf{v}' - \mathbf{v}|^2).$$

But

$$|\mathbf{v}'_* - \mathbf{v}|^2 = |\mathbf{v}_* - \mathbf{v} - \mathbf{nn}\cdot(\mathbf{v}-\mathbf{v}_*)|^2 \leqslant 4|\mathbf{v}-\mathbf{v}_*|^2,$$
$$|\mathbf{v}' - \mathbf{v}|^2 = |\mathbf{nn}\cdot(\mathbf{v}-\mathbf{v}_*)|^2 \leqslant |\mathbf{v}-\mathbf{v}_*|^2.$$

Hence we have

$$|\psi(\mathbf{v}') + \psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v})| \leqslant 54K|\mathbf{v}_* - \mathbf{v}|^2$$

and the statement follows.

In the case of a bounded kernel, we know that

$$\int_0^t \int_{S_2} \int_0^1 \int_{R^3} \int_{R^3} f(\mathbf{v}_*) f(\mathbf{v}) |\mathbf{v}_* - \mathbf{v}|^2 d\mu\, dt$$

is uniformly bounded. Hence we can conclude that the collision term in a weak form is uniformly bounded. Actually, we have shown a slightly more general result.

Lemma 2.5. Under the above assumptions

$$\int_0^t \int_{S_2} \int_0^1 \int_{R^3} \int_{R^3} \int_{\Omega^2} ff(\mathbf{v}) |\psi(\mathbf{v}') + \psi(\mathbf{v}'_*) - \psi(\mathbf{v}_*) - \psi(\mathbf{v})| B(|\mathbf{v}_* - \mathbf{v}|, \theta) d\,\mu dt$$

is bounded.

3. EXISTENCE OF WEAK SOLUTIONS

We will now show that the estimates from Section 2 imply the existence of a global weak solution for the initial value problem. We could do this by extracting convergent subsequences from sets of solutions of suitably truncated approximating problems, as done in the famous paper[(6)], but in the process we would have to repeat most of the estimates done there. The approach we shall take instead is to use the knowledge that there is a renormalized solution in the sense of DiPerna–Lions. We shall argue that the estimates from the previous sections entail that this solution is indeed a weak solution in the sense defined in Section 1.

Theorem 3.1. Let $f_0 \in L^1(\mathbf{R} \times \mathbf{R}^3)$ be such that

$$\int f_0(\cdot)(1 + |x|^2 + |\mathbf{v}|^2) d\mathbf{v}\, dx < \infty; \qquad \int f_0 |\ln f_0(.)| d\mathbf{v}\, dx < \infty. \quad (3.1)$$

Also, assume that the collision kernel B satisfies the conditions made in Section 2. Then there is a weak solution $f(x, \mathbf{v}, t)$ of the initial value problem (1.1), (1.4), such that $f \in C(\mathbf{R}_+, L^1(\mathbf{R} \times \mathbf{R}^3))$, $f(., 0) = f_0$. This solution conserves energy globally.

Proof. We use the results for collision term cutoff at small relative speeds[4]; because of the estimate of the previous section we can remove the cutoff and conclude that we have a solution of the Boltzmann equation in weak form for f. Conservation of energy is obvious. ∎

4. CONCLUDING REMARKS

We have proved existence of a weak solution of the nonlinear Boltzmann equation, without truncation on the collision kernel for small relative speeds, in the one-dimensional case. The solution conserves energy globally.

5. ACKNOWLEDGMENTS

A preliminary form of the main result of this paper was obtained in 1998; a similar result, obtained by a different method, is contained in Cedric Villani's thesis[8]. At that time, I exchanged information with him; see a remark in page 416 of his thesis. The result of this exchange was that neither of us published his results, because each of us thought that the other had a better result.

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